

ANALYSIS OF LAMINATED COMPOSITE PLATES
USING FINITE ELEMENT METHOD

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Abstract

With a view to obtain the optimum design of composite plates, free vibration and static analysis has been made using finite element displacement formulation. The plate consists of number of layers of orthotropic material. The lamination is assumed to be symmetric about the middle surface of the plate. The stiffness and mass matrices for a rectangular plate bending element with twelve degrees of freedom have been obtained. The free vibration analysis leads to a generalized eigenvalue problem which has been solved by subspace iteration technique.

Effect of orientation of fibres in a composite plate on the natural frequencies and corresponding mode shapes has been studied. Also, the effect of square cut-out in a square plate on the natural frequencies and mode shapes has been obtained. The results are presented in a graphical form and comparisons are made with available data in literature.

I. Introduction

Composite materials, because of their high strength-to-weight and stiffness-to-weight ratio, are better suited as structural elements for light weight structures like aircrafts and spacecrafts over conventional metal construction. In the case of fibrous composites, the properties of a structural element can be varied by changing the orientation of the fibres in the matrix without any change in the amount of material consumed. Thus, when composite materials are used, one should look for an optimal configuration by selecting proper orientation of the fibres and matrix to fibre volume ratio, such that the resulting structure satisfies all the service criteria.

In addition to static loads, most structures are subjected to dynamic loads as well. Hence, analysis of natural vibration plays an important role in design of any structure. Present paper aims at obtaining the natural frequencies and mode shapes of layered plates of composite fibrous materials using finite element method (displacement formulation). Pryor⁽⁷⁾, Mawenya⁽⁴⁾, Hinton⁽²⁾ and others have presented the analysis of composite plates using various finite element displacement

models. Srinivas and Rao⁽⁸⁾ and Srinivas et. al⁽⁹⁾ have investigated the free vibration of laminated plates using Mindlin's⁽⁵⁾ plate theory. They have obtained a series solution for the governing differential equation. Although some data is available in literature for fundamental frequencies of plates of various shapes, there appears to be very little report regarding higher frequencies, corresponding mode shapes and also the effect of fibre orientation on the natural frequencies and mode shapes in a composite plate. Service requirements often necessitates introduction of cut-outs in plates and hence it is necessary to determine the effect of cutouts on natural frequencies of such plates.

In the present study, a well known and extensively used plate bending element was extended for laminated composite plates of symmetric cross-section. Stiffness and consistent mass matrices were derived for rectangular plate elements and were used to obtain the natural frequencies and mode shapes of rectangular plates with and without cut-outs. Results are presented in graphical form showing the effect of fibre orientation and cut-out size on the natural frequencies.

II. Finite Element Formulation

A laminated composite plate consists of an arbitrary number of bonded plies. For symmetric laminates, where the cross section is symmetric about the middle surface both in geometry and material properties, there exists no coupling between bending and membrane actions and hence only bending action need to be considered for transverse oscillations.

For finite element idealization of laminated plates, a rectangular non-conforming element with four corner nodes and three degrees of freedom per node⁽¹⁰⁾ has been chosen. The degrees of freedom at a typical node 'i' are the transverse displacement w_i and the rotations θ_{x_i} and θ_{y_i} about x and y axes respectively. The positive sense of rotations is taken to be counter-clockwise. The nodal degrees of freedom vector $\{\delta_i\}$ for the 'i'th node is given by

$$\{\delta_i\} = [w_i \ \theta_{x_i} \ \theta_{y_i}]^T \quad (1)$$

The element degrees of freedom vector $\{\delta^e\}$ is obtained by combining the nodal degrees of freedom vector of the four nodes as follows :

$$\{\delta^e\} = [\{\delta_1\}^T \{\delta_2\}^T \{\delta_3\}^T \{\delta_4\}^T]^T \quad (2)$$

The displacement field (transverse displacement of the middle surface of the plate) is assumed to be a fourth order polynomial in terms of twelve parameters given by⁽¹⁰⁾,

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \quad (3)$$

$$= [P(x,y)] \{\alpha\} \quad (4)$$

$$[P(x,y)] = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y & xy^2 & y^3 & x^3 y & xy^3 \end{bmatrix} \quad (5)$$

and

$$\{\alpha\} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \dots \ \alpha_{11} \ \alpha_{12}]^T \quad (6)$$

The displacement field can be expressed in terms of the element degrees of freedom vector $\{\delta^e\}$ by writing twelve equations linking the displacements and slopes at all the four nodes of the elements in term of $\{\alpha\}$. Thus, for a typical node 'i', we have

$$w_i = w(x_i, y_i) = [P(x_i, y_i)] \{\alpha\} \quad (7)$$

$$\theta_{x_i} = \frac{\partial w(x_i, y_i)}{\partial y} = [P_{,y}(x_i, y_i)] \{\alpha\} \quad (8)$$

$$\theta_{y_i} = -\frac{\partial w(x_i, y_i)}{\partial x} = -[P_{,x}(x_i, y_i)] \{\alpha\} \quad (9)$$

where ',' denotes differentiation. Equations (7), (8) and (9) can be written for all four nodes and the resulting twelve equations can be expressed in the matrix form as

$$\{\delta^e\} = [C] \{\alpha\} \quad (10)$$

Combining equations (4) and (10), the displacement field can now be expressed in terms of $\{\delta^e\}$ as

$$w(x,y) = [\bar{N}] \{\delta^e\} \quad (11)$$

where

$$[\bar{N}] = [P] [C]^{-1} \quad (12)$$

The generalized strains for a plate bending problems are the curvatures k_x , k_y and k_{xy} . These can be expressed in terms of $\{\alpha\}$ as follows :

$$k_x = -\frac{\partial^2 w}{\partial x^2} = -[P_{,xx}] \{\alpha\} \quad (13)$$

$$k_y = -\frac{\partial^2 w}{\partial y^2} = -[P_{,yy}] \{\alpha\} \quad (14)$$

$$k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = -2[P_{,xy}] \{\alpha\} \quad (15)$$

Equations (13), (14) and (15) can be combined and written in the matrix form as

$$\{k\} = [\bar{B}] \{\alpha\} \quad (16)$$

where

$$\{k\} = [k_x \ k_y \ k_{xy}]^T \quad (17)$$

The material matrix $[D]$ for plate bending problem relates the moment vector $\{M\}$ to the curvature vector $\{k\}$ through the relation

$$\{M\} = [D] \{k\} \quad (18)$$

where

$$\{M\} = [M_x \ M_y \ M_{xy}]^T \quad (19)$$

In case of laminated composite plate, the matrix $[D]$ can be shown to be⁽³⁾,

$$[D] = \sum_{i=1}^n \frac{z_{i+1}^3 - z_i^3}{3} [\bar{Q}^i] \quad (20)$$

where n is the total number of layers and z_i is the z-coordinate of the bottom surface of the ith layer as shown in Fig. 1.

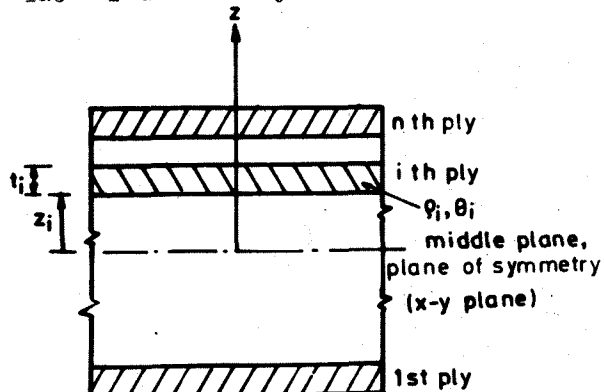


Fig. 1. Cross Section of Symmetric Laminated Composite Plate

The matrix $[\bar{Q}^i]$ relates the stresses and strains in x-y coordinate system of the ith layer according to the equation

$$\{\sigma^i\} = [\bar{Q}^i] \{\epsilon^i\} \quad (21)$$

where

$$\{\sigma^i\} = \begin{bmatrix} \sigma_x^i & \sigma_y^i & \tau_{xy}^i \end{bmatrix}^T \quad (22)$$

and

$$\{\epsilon^i\} = \begin{bmatrix} \epsilon_x^i & \epsilon_y^i & \nu_{xy}^i \end{bmatrix}^T \quad (23)$$

The superscript 'i' in equations (21), (22) and (23) refers to ith layer. The elements of the matrix $[\bar{Q}^i]$ depends on the material properties of the ith layer and also on the orientation of the fibres in ith layer with respect to x-y coordinate system. The explicit expression for $[\bar{Q}^i]$ can be found in Ref. 3.

The element stiffness matrix $[k_e]$ can now be expressed as⁽¹⁰⁾

$$[k_e] = ([C]^{-1})^T \left(\iint_A [\bar{B}]^T [D] [\bar{B}] dA \right) [C]^{-1} \quad (24)$$

The element mass matrix $[m_e]$ is given by

$$[m_e] = ([C])^T \left[\iint_A [P]^T [P] dA + \alpha \left(\iint_A [P_{,x}]^T [P_{,x}] dA + \iint_A [P_{,y}]^T [P_{,y}] dA \right) \right] [C]^{-1} \quad (25)$$

where

$$\alpha = \sum_{i=1}^n \left(\frac{z_{i+1}^3 - z_i^3}{3} \right) \rho_i \quad (26)$$

and

$$\beta = \sum_{i=1}^n (z_{i+1} - z_i) \rho_i \quad (27)$$

The integrations in equations (24) and (25) are to be carried out over the entire area of the element.

The element stiffness and mass matrices are then 'assembled'⁽¹⁰⁾ to form plate stiffness matrix $[K]$ and mass matrix $[M]$.

The natural frequency and corresponding mode shape of free vibration can now be obtained by solving the generalized algebraic eigenvalue problem⁽¹⁰⁾

$$[K] \{A\} = \omega^2 [M] \{A\} \quad (28)$$

where, ω is the natural frequency in radians per second and $\{A\}$ is the global displacement vector which gives the mode

shape of free vibration. In the present study, the first few eigenvalues and corresponding eigenvectors were obtained by method of sub-space iteration⁽¹⁾.

III. Results and Discussion

Natural frequencies were obtained for rectangular laminated composite plate of symmetric cross-section and square plates (isotropic, orthotropic and laminated) with square cut-outs. Material properties assumed for obtaining numerical results are (i) Isotropic-with $\nu=0.3$, and (ii) Glass Epoxy (orthotropic)-with $E_1/E_2=3.0$, $G_{12}/E_2=0.5$, $\nu_{12}=0.25$.

The finite element solution for fundamental frequency of a simply supported specially orthotropic plate was compared with available closed form solution⁽³⁾. The relative error was found to reduce from 5 to 1 percent as the number of elements were increased from 9 to 64. Similarly, the error in static deflection of a simply supported square isotropic plate with concentrated load at the centre was found to be 2 percent for 25 element solution.

Rectangular Plates

Rectangular plates under consideration has an aspect ratio 1.5. The plates consist of three layers of Glass-Epoxy of equal thickness with fibres in the central layer parallel to the longer edge. Figs. 2 and 3 show the variation of normalized natural frequencies of simply supported and clamped plates respectively with change in θ_1 , where θ_1 is the orientation of the fibres in top and bottom layers measured from the longer edge of the plate. Also shown in the figures are the corresponding mode shapes at $\theta_1=0^\circ$ and $\theta_1=90^\circ$.

It may be observed from Figs. 2 and 3 that, not only do the frequencies change with change in orientation of the fibres, but the corresponding mode shapes also change continuously with change in θ_1 . For example, the mode shape corresponding to $m=2, n=2$ at $\theta_1=0^\circ$ changes continuously with change in θ_1 and becomes one corresponding to $m=3, n=1$ at $\theta_1=90^\circ$ and vice-versa, both for the clamped and the simply supported plates. Similarly, in Fig. 3, it may be observed that the mode shape corresponding to $m=3, n=2$ at $\theta_1=0^\circ$ changes continuously with change in θ_1 and assumes a form corresponding to $m=4, n=1$ at $\theta_1=90^\circ$. An additional feature to be observed from these graphs is the intersection of the curves, which indicate that the order of the frequencies

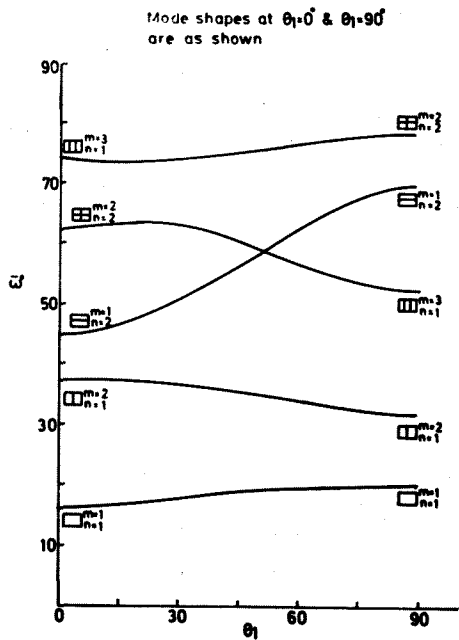


Fig. 2. Variation of $\bar{\omega}^0$ with θ_1 .
(Rectangular Simply Supported Laminated Plate; Aspect Ratio 1.5)

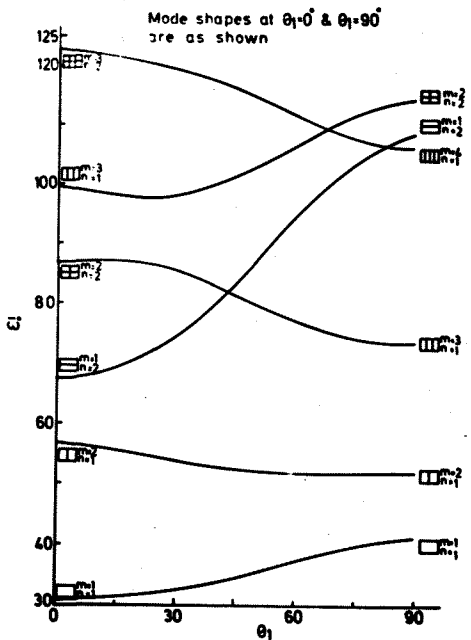


Fig. 3. Variation of $\bar{\omega}^0$ with θ_1 .
(Rectangular Clamped Laminated Plate; Aspect Ratio 1.5)

corresponding to two distinct mode shapes may interchange with a change in fibre orientation. It is observed from these figures that, in general, the change in magnitude of natural frequencies with a variation in fibre orientation, is higher when the fibres are oriented at approximately 45° to the edges compared to the change, when the fibres are nearly parallel to the edges. For simply supported and clamped plates of same cross section and aspect ratio 1, 1.5, 2, 2.5, similar feature was also observed.

Plates with Cut-outs

In order to study the effect of centrally placed square cut-out in simply supported square plates on the natural frequencies of vibration, plates of different cross sections were considered. The variation of natural frequencies with change in cut-out parameter, where cut-out parameter is defined as the ratio of length of cut-out (b) to length of the plate (a), are shown in Figs. 4, 5, 6 and 7. Fig. 4 refers to an isotropic plate. Figs. 5 and 6 refer to orthotropic plates of Glass-Epoxy with fibres oriented at 0° and 45° respectively with an edge. Fig. 7 refers to a laminated plate consisting of three layers of Glass-Epoxy of equal thickness. The fibres in the central layer are parallel to an edge and those in the top and bottom layers are oriented at 45° with the same edge. Also shown in these figures are the corresponding mode shapes at $b/a = 0.0$ (plate without cut-out).

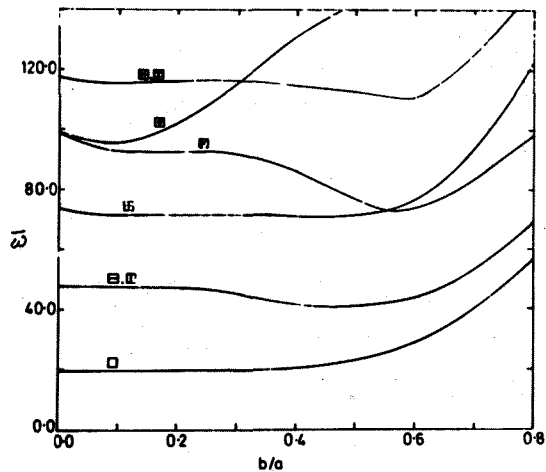


Fig. 4. Variation of $\bar{\omega}$ with cut-out parameter (Isotropic plate)

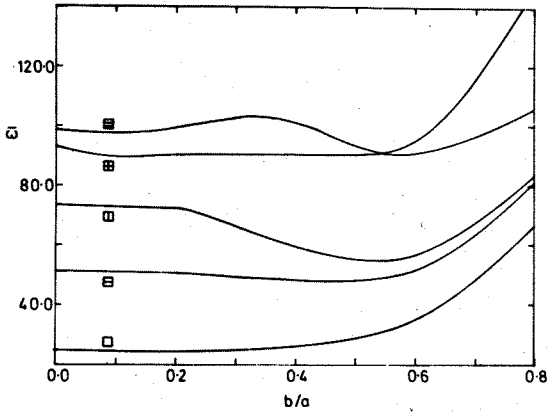


Fig. 5. Variation of $\bar{\omega}$ with cut-out parameter (Orthotropic plate, Fibres parallel to an edge)

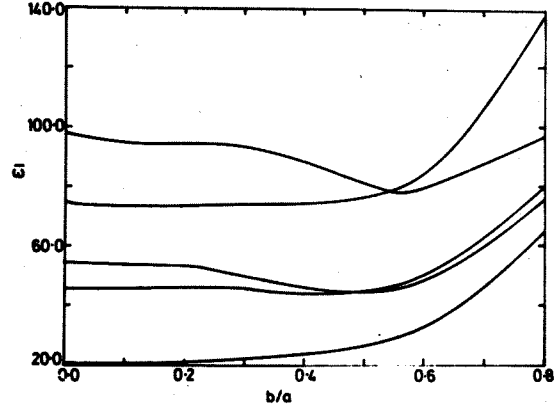


Fig. 7. Variation of $\bar{\omega}$ with cut-out parameter (Laminated plate)

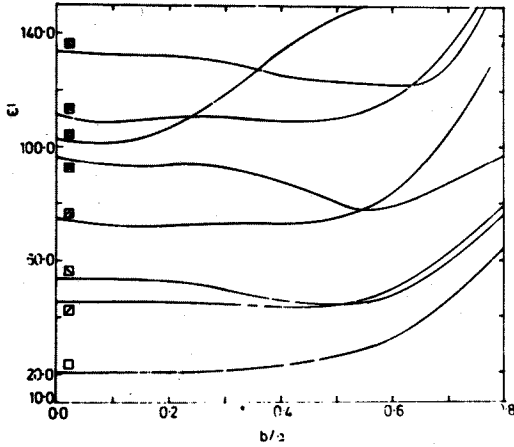


Fig. 6. Variation of $\bar{\omega}$ with cut-out parameter (Orthotropic plate, Fibres at 45° with edges)

In these figures too, we observe the intersection of curves corresponding to two distinct mode shapes which indicate that the order of frequencies corresponding to two distinct mode shape may get interchanged with change in cut-out dimension. Natural frequencies are, in most cases, found to remain fairly constant for small cut-outs, that is, for cut-out parameter between 0 and 0.3. As the cut-out dimension is increased beyond this limit, the change in natural frequencies become appreciable. For those mode shapes corresponding to which the frequency decreases with increase in cut-out

dimension, minimum frequency is found to occur when the cut-out parameter is approximately equal to 0.5. Beyond this limit, the frequencies corresponding to almost all the mode shapes are found to increase. The mode shapes also deform with change in cut-out dimension.

For isotropic square plates without any cut-out, the frequencies corresponding to mode shapes with $m = 3, n = 1$ and $m = 1, n = 3$ are identical. Hence any linear combination of these mode shapes is also a mode shape of natural vibration with same frequency. Two such linear combinations, which are also orthogonal to each other are (i) a mode shape with circular nodal line and (ii) a mode shape with diagonal nodal lines. On introduction of a square cut-out in the plate, it is observed from Fig. 4 that, two modes of vibration with circular nodal line and diagonal nodal lines exist with distinct frequencies. Mode shape corresponding to $m = 3, n = 1$ or $m = 1, n = 3$ no longer exist for plates with cut-out. It is also observed that, whereas the frequency corresponding to a mode shape with circular nodal line increases with increase in cut-out dimension, that corresponding to a mode shape with diagonal nodal lines decreases with increase in cut-out dimension.

In Table 1, the finite element solution for fundamental frequency of isotropic square plate with a square cut-out is compared with the results presented in Ref. 6.

IV. Conclusions

In this paper, an attempt was made to study the natural vibration of composite, orthotropic and isotropic plates with and without cut-outs.

The results presented in the previous section indicate that, not only do the natural frequencies vary with variation of different parameters, but the corresponding mode shapes vary as well. Hence, in a design problem involving forced vibration, it is not sufficient to just separate the natural frequency and the frequency of the forcing function by suitable choice of a parameter, but the corresponding change in the mode shape must be taken into account in choosing a proper parameter. For example, a structure supporting a plate or a mechanism supported on a plate are often designed by taking into consideration the mode shapes of vibration. Thus a change in any plate parameter may necessitate a redesign of entire structural system.

In case of composite plates, it is seen that the matrix $[D]$, appearing in the expression of stiffness matrix, changes with a change in orientation of the fibres. All the other matrices that are used for determination of stiffness and mass matrices are invariant to the change in fibre orientation. Hence, it is expected that there may exist a correlation between the change in the matrix $[D]$ and the change in mode shapes and frequencies. Some more study has to go into this aspect of the problem.

Table 1

Comparison of non-dimensional frequency of a simply supported isotropic plate

Cut-out parameter	0.0	0.167	0.333	0.5
Ref. 6 F.E.M. Solution	19.63	19.48	21.45	26.05
	19.6	19.6	20.00	22.00

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Notations

- a - Length of a square plate.
- b - Length of a square cut-out in a square plate.
- D_{22} - Element of matrix $[D]$ belonging to second row and second column.
- D_{22}^0 - Value of D_{22} with $\theta_i = 0$ in all layers.
- l_y - Length of shorter side of a rectangular plate (parallel to y-direction).
- θ_i - Orientation of fibres in i th layer measured from x-axis (parallel to longer side of the plate).
- ρ_i - Density of material in i 'th layer.
- f_i - Natural frequency in radians per second.
- $\bar{\omega}$ - $\frac{\omega}{\sqrt{\frac{D_{22}}{\beta l^4 y}}}$
- ω^0 - $\frac{\omega}{\sqrt{\frac{D_{22}^0}{\beta l^4 y}}}$